A NOTE ON THE WAVE FLOW OF THIN LIQUID LAYERS ON A VERTICAL SURFACE

L. P. Kholpanov

Inzhenerno-Fizicheskii Zhurnal, Vol. 8, No. 4, pp. 488-492, 1965

Periodic flow of thin liquid layers has been investigated with allowance for the shear stress at the liquid-gas interface. The results indicate that the flow friction of two-phase systems is an important factor in calculating the very characteristic parameters a_0, λ of wave motion of a liquid film.

Wave flow of thin liquid films has been studied in a number of interesting papers, whose results are widely used in the analysis of film heat and mass transfer [1-8]. In this paper the problem is analyzed by Kapitsa's method [2], as im-by Bushmanov [4]. Within the limits of the approximation used, we have eliminated the shortcomings of the boundary conditions adopted in [4].

It is known that the wave flow of thin liquid layers exposed to the influence of gravity and a moving gas is described by the nonstationary Navier-Stokes equations. In the approximations of the Prandtl boundary theory, these equations take the form

$$\partial u_x/\partial t + u_x \partial u_x/\partial x + v_y \partial u_x/\partial y = -\partial p/\rho\, \partial x + g + \nu\, \partial^2 u_x/\partial y^2, \tag{1}$$

$$\partial u_x/\partial x + \partial v_y/\partial y = 0. \tag{2}$$

Following the method of [2], we may put Eq. (1) in the form

$$(u_x - k)\, \partial u_x/\partial x + v_y \partial u_x/\partial y = -\partial p/\rho\, \partial x + g + \nu\, \partial^2 u_x/dy^2. \tag{3}$$

Averaging all terms of (3) with respect to y gives

$$\overline{(u_x - k)\, \partial u_x/\partial x} + \overline{v_y \partial u_x/\partial y} = -\partial p/\rho\, \partial x + g + \overline{\nu\, \partial^2 u_x/\partial y^2}. \tag{4}$$

In the flow under examination, the total pressure gradient consists of a component due to the moving gas and a component due to surface tension forces. For thin films the pressure gradient is

$$\partial p/\partial x = \psi - \sigma\, \partial^3 a/\partial x^3. \tag{5}$$

Putting $G = g - \Psi/\rho$, and taking account of (5), we reduce (4) to the form

$$\overline{(u_x - k)\, \partial u_x/\partial x} + \overline{v_y \partial u_x/\partial y} = G + \sigma\, \partial^3 a/\rho\, \partial x^3 + \overline{\nu\, \partial^2 u_x/\partial y^2}. \tag{6}$$

Let us specify the velocity by means of the following polynomial

$$u_x = Ay^2 + By + C, \tag{7}$$

where A, B, and C are periodic coefficients and functions of $f(x - kt)$. Their values are determined from the boundary conditions:

$$\text{when } y = 0 \quad u_x = 0; \tag{8}$$

$$\text{when } y = a \quad \mu\, \partial u_x/\partial y = \tau_0. \tag{9}$$

Condition (9) indicates that a shear stress acts at the gas-liquid interface. Combining (8) and (9) with (7), we find the velocity

$$u_x = A(y^2 - 2ay) + \tau_0 y/\mu. \tag{10}$$

We express A in terms of the mean velocity of the liquid film

$$u = \int_0^a u_x dy/a = \int_0^a (Ay^2 + \tau_0 y/a - 2Aay)\, dy/a \tag{11}$$

or

$$A = -3u(x, t)/2a^2(x, t) + 3\tau_0/4a(x, t)\mu. \tag{12}$$

Finally, after substituting (12) into (10), we obtain an expression for the velocity

$$u_x = (3\tau_0/4a\,\mu - 3u/2a^2)(y^2 - 2ay) + \tau_0\, y/\mu. \tag{13}$$

To find v_y we use the equality

$$v_y = -\int_0^y (\partial u_x/\partial x)\, dy + f(x). \tag{14}$$

The function $f(x)$ is equal to zero when $y = 0$, since $v_y = 0$ at the wall. Substituting into (14) the value of $\partial u_x/\partial x$ found using (13), we obtain

$$v_y = -(3y^2/2a^2)\,[(\dot{\bar{u}}a - \bar{u}\dot{a}) - (\dot{\bar{u}}a - 2\bar{u}\dot{a})\,y/3a] + \tau_0\,\dot{a}y^3/4\mu\, a^2. \tag{15}$$

Omitting intermediate transformations associated with finding $\partial u_x/\partial x$, $v_y \partial u_x/\partial y$, $\partial^2 u_x/\partial y^2$ with the aid of (13) and (15), we put (6) in the form:

$$\sigma\, \dddot{a}/\rho + G - (1/10a)\,[3\bar{u}\,(3\dot{\bar{u}} - \bar{u}\dot{a}) - 5k\,(2\dot{\bar{u}}a - \bar{u}\dot{a})] + (\tau_0/20\mu)(3\bar{u}\dot{a} + $$
$$+ 4\dot{\bar{u}}a - 5k\dot{a}) - 3\nu\,\bar{u}/a^2 - \tau_0^2\, a\dot{a}/40\mu + 3\tau_0\nu/2a\mu. \tag{16}$$

In accordance with the continuity equation,

$$\bar{u} = \bar{u}_0\,[Z - (Z - 1)\,a_0/a],\quad Z = k/u_0. \tag{17}$$

Considering waves of small amplitude, we write the layer thickness a in the form

$$a = a_0(1 + \varphi). \tag{18}$$

Taking (17) and (18) into account, we reduce (16), after lengthy transformations, the final form:

$$(\sigma a_0\, \dddot{\varphi}/\rho + G)\,(1 + \varphi)^3 + \dot{\varphi}\,\{\bar{u}_0^2\,(Z^2 - 2.4Z + 1{,}2) +$$
$$+ [(1 + \varphi)^3\, \tau_0\, u_0 a_0/20\mu]\,[(Z - 1)/(1 + \varphi) - 2Z] -$$
$$- (\tau_0^2\, a_0^2/40\mu^2)\,(1 + \varphi)^4\,\} - 0.4\,\bar{u}_0^2\, \dot{Z}_0^2\varphi\varphi - 0.2\bar{u}_0^2\, \dot{Z}^2\, \varphi\varphi^2 -$$
$$- (3\nu\, u_0/a_0^2)\,(1 + Z\varphi) + 3\tau_0\nu\,(1 + \varphi)^2/2a_0\,\mu = 0. \tag{19}$$

Assuming the amplitude of function φ and its derivatives to be small quantities, we shall examine (19) and its solutions in the zeroth and first approximations. We can obtain the zeroth approximation, or absence of wave motion, from (19), by setting φ and its derivatives equal to zero:

$$G - 3\nu\,\bar{u}_0/a_0^2 + 3\,\tau_0\nu/2\mu\, a_0 = 0. \tag{20}$$

If we assume

$$\psi = N\,\Delta\, p_{tr}/l,\quad \tau_0 = \mp\,\Delta\, p_{tr} a_0/l, \tag{21}$$

where the "minus" and "plus" signs refer, respectively, to reverse and forward flow, and where N is a proportionality factor, then the film thickness may be expressed by the equation

$$a_0^2 = 3\nu\,\bar{u}_0/g\,[1 \mp A\,(3/2 - N)], \tag{22}$$

where the "minus" and "plus" signs refer, respectively, to reverse and forward flow. If the gravity forces are much greater than those expended on trickling, i.e., $l\gamma_* \gg \Delta p_{tr}$, then we obtain the widely known formula for determining the layer thickness. We also obtain this formula if the proportionality factor is 3/2.

In (19) we retain only terms containing φ and its derivatives in powers not exceeding the first:

$$\sigma\, a_0\dddot{\varphi}/\rho + 3G\varphi + \dot{\varphi}\,[\bar{u}_0^2\,(Z^2 - 2.4Z + 1.2) -$$

$$-\tau_0 \bar{u}_0 a_0 (Z+1)/20\mu - \tau_0^2 a_0^2/40\mu^2] -$$ (23) (cont'd)

$$-3\nu \bar{u}_0 Z\varphi/a_0^2 + 3\tau_0\nu\varphi/\mu a_0 + G - 3\nu\bar{u}_0/a_0^2 + 3\tau_0\nu/2\mu a_0 = 0.$$

For existence of a steady periodic solution, the constant term and the coefficient of φ must be equal to zero, i.e.,

$$G - \nu\bar{u}_0 Z/a_0^2 + \tau_0\nu/\mu a_0 = 0.$$

With (21) taken into account, the layer thickness may be expressed as:

$$a_0^2 = Z\nu\bar{u}_0/g\ [1 \mp A(1-N)], \tag{24}$$

where the "minus" and "plus" signs refer, respectively, to reverse and forward flow. If we assume, in particular, that the proportionality factor N equals unity, or if $l\gamma_* \gg \Delta p_{tr}$, then (24) goes over into the formula obtained by Kapitsa [2]. Comparing (22) and (24), we obtain

$$Z = 3[1 \mp A(1-N)][1 \mp A(3/2-N)].$$

With these assumptions, (23) takes the form:

$$\sigma a_0 \dddot{\varphi}/\rho + \dot{\varphi}\,[\bar{u}_0^2\,(Z^2 - 2.4Z + 1.2) - -\tau_0\bar{u}_0 a(Z+1)/20\mu - \tau_0^2 a_0^2/4\mu^2] = 0. \tag{25}$$

The stable periodic solution, according to (25), has the form

$$\varphi = \alpha \sin n(x - kt),$$

where the wave number is given by

$$n^2 = \rho\bar{u}_0^2\,(Z^2 - 2.4Z + 1.2)/\sigma a_0 - -\tau_0 u_0 a_0 \rho (Z+1)/20\mu\sigma a_0 - \tau_0^2 a_0^2 \rho/40\mu^2\sigma a_0.$$

Hence the wavelength is given by the expression:

$$\lambda = \lambda_K/[1 - \vartheta(Z+1)/20(Z^2 - 2.4Z + 1.2) - -\vartheta^2/40\,(Z^2 - 2.4Z + 1.2)]^{1/2}, \tag{26}$$

where $\lambda_K = (2\pi/\bar{u}_0)\,[\sigma a_0/\rho\,(Z^2 - 2{,}4Z + 1{,}2)]^{1/2}$ with the correction of [4], and

$$\vartheta = \tau_0 a_0/\mu u_0. \tag{27}$$

Taking (21) and (24) into account, (27) may be written as:

$$\vartheta = \mp A/[1 \mp A(1-N)]. \tag{28}$$

Thus the wavelength, in addition to its dependence on other factors revealed by Kapitsa's data, also depends on the flow friction of the two-phase system. It follows from (26) that the wavelength as a function of flow friction passes through extrema. To determine these, the wavelength in (26) must be differentiated with respect to ϑ, and the result equated to zero. We obtain

$$\vartheta_e = -(Z+1).$$

Taking (28) into account, we shall write down the value of the maximum and minimum, respectively: for reverse flow

$$\vartheta_{max} = A/[1 - A(1-N)] = Z + 1,$$

and for forward flow

$$\vartheta_{min} = A/[1 + A(1-N)] = -(Z+1).$$

The formulas obtained permit an approximate evaluation of wavelength without any experimental constants. Thus, substituting ϑ_{max} and ϑ_{min} into (26), we obtain:

$$\lambda = \lambda_K/[1 - (Z+1)^2/40\,(Z^2 - 2.4Z + 1.2)]^{1/2}, \tag{29}$$

for forward flow

$$\lambda = \lambda_K/[1 + (Z+1)^2/40(Z^2 - 2.4Z + 1,2)]^{1/2}. \tag{30}$$

Using values of ϑ_e, we find the thickness of the film of liquid running off along a vertical surface in wave flow: for reverse flow

$$a_0^2 = Z(Z+1)\nu\bar{u}_0/Ag, \tag{31}$$

for forward flow

$$a_0^2 = -Z(Z+1)\nu\bar{u}_0/Ag. \tag{32}$$

The results indicate the special features of the wave process of liquid runoff and, in particular, show that the flow friction of a two-phase flow is important for the calculation of the parameters a_0 and λ, which are very characteristic for the wave motion of a liquid film. These are given by (26), (31), and (32). While our analysis does not complete the solution of this very complex problem, it does indicate one possible method of examining it.

NOTATION

a – film thickness; ψ – gradient due to gas flow; σ – surface tension at interface; τ_0 – constant shear stress at interface; u_0 – mean flow velocity of liquid at mean cross section; a_0 – mean film thickness; $a_0\varphi$ – deviation from mean thickness at surface; λ_K – wavelength determined from Kapitsa's data; subscript e – extremum.

REFERENCES

1. P. A Semenov ZhTF, no. 7, 8, 1944.
2. P. L. Kapitsa, ZhETF, no. 3, 1948.
3. P. L. Kapitsa and S. P. Kapitsa, ZhETF, no. 2, 3, 1949.
4. V. K. Bushmanov, Uch. zap. Ivanovskogo gos. ped. in-ta, 18, 217, 1958.
5. B. S. Kasimov and F. F. Zigmund, no. 4, 1963.
6. B. I. Konobeev, V. A. Malyusov and N. M. Zhavoronkov, Khimicheskaya promyshlennost, no. 7, 1961.
7. B. I. Konobeev, V. A. Malyusov and N. M. Zhavoronkov, Khimicheskaya promyshlennost, no. 3, 1957.
8. N. M Zhavoronkov, The Hydraulic Basis of the Scrubbing Process and Heat Transfer in Scrubbers [in Russian], Izd-vo "Sovetskaya nauka," 1944.
9. A. F. Andreev, ZhETF, 755, 1963.

28 May 1964

Polytechnic Institute, Tula